Subdivision of A-Splines

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What is an A-spline?

- “Algebraic spline”
- Implicit polynomial curve $f(x, y) = 0$ in compact domain.
- We’ll focus on planar, *triangular* A-splines.

Motivation

- Quickly draw curves and surfaces
- Implicit form easy to derive, manipulate
- Implicit form hard to render
- Until I show you what I got…

The Problem

- Given description of image A
- Find:
  - Initial set A_0
  - Function S
- Such that $A_0, S(A_0), S^2(A_0), \ldots \xrightarrow{\text{quickly}} A$.

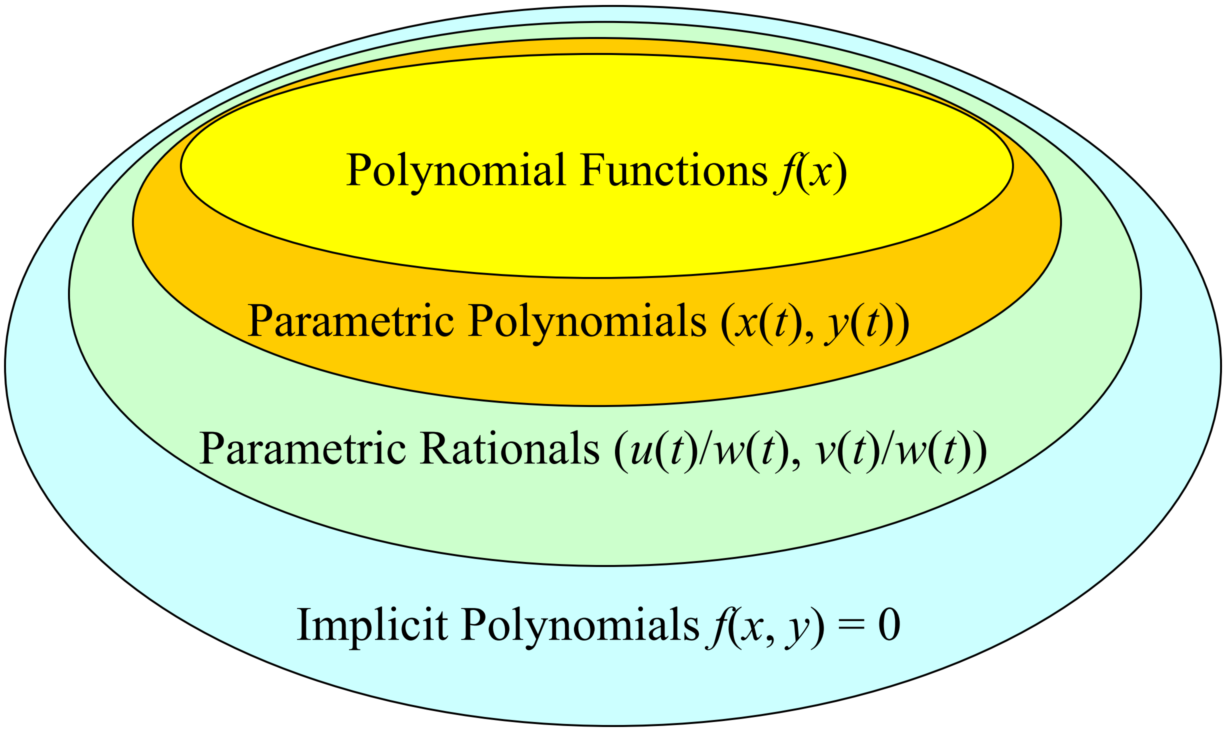
The Solution: Subdivision

- Start with A_0 = a "control polygon"

- Subdivision function S splits polygons into subpolygons.

- Define S so that convergence is fast.

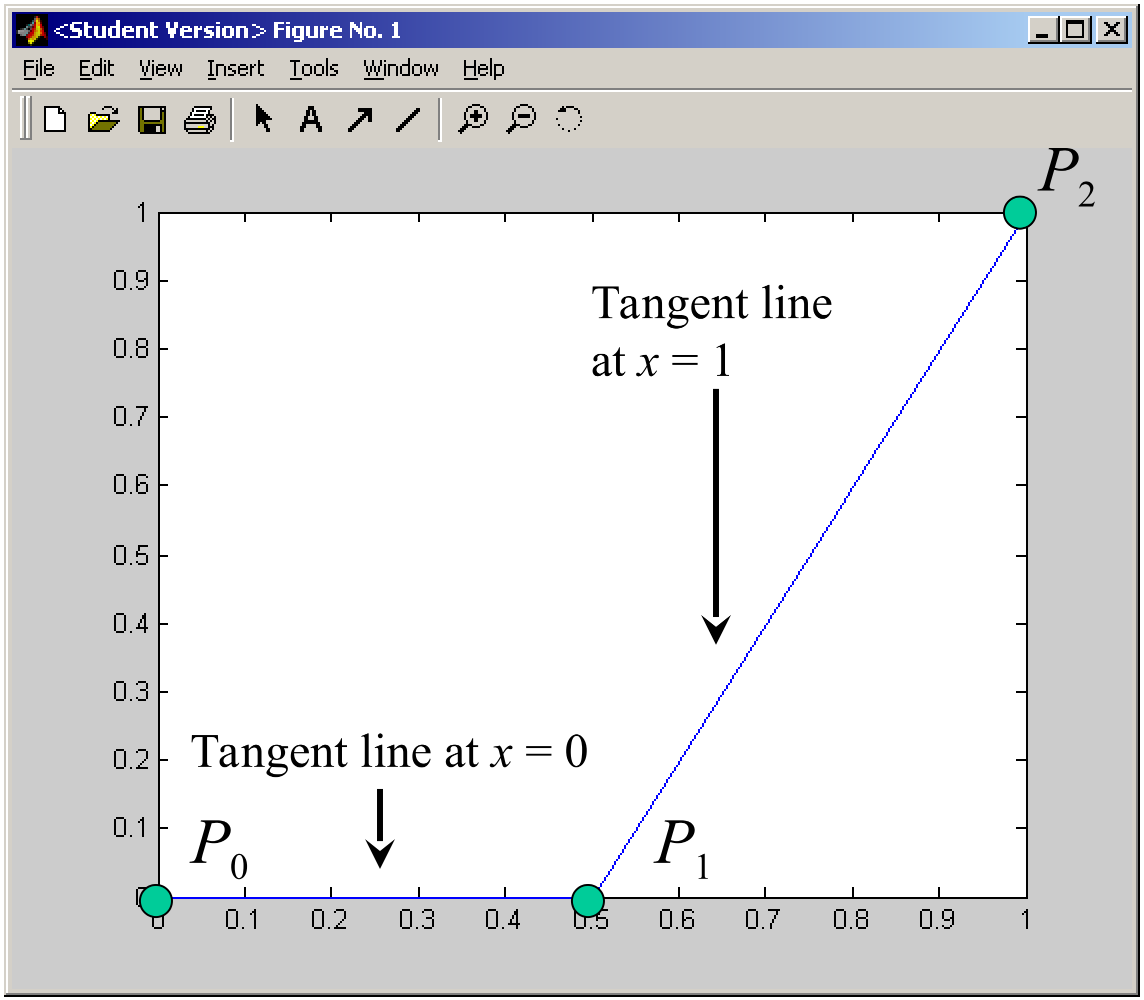
What can we subdivide?



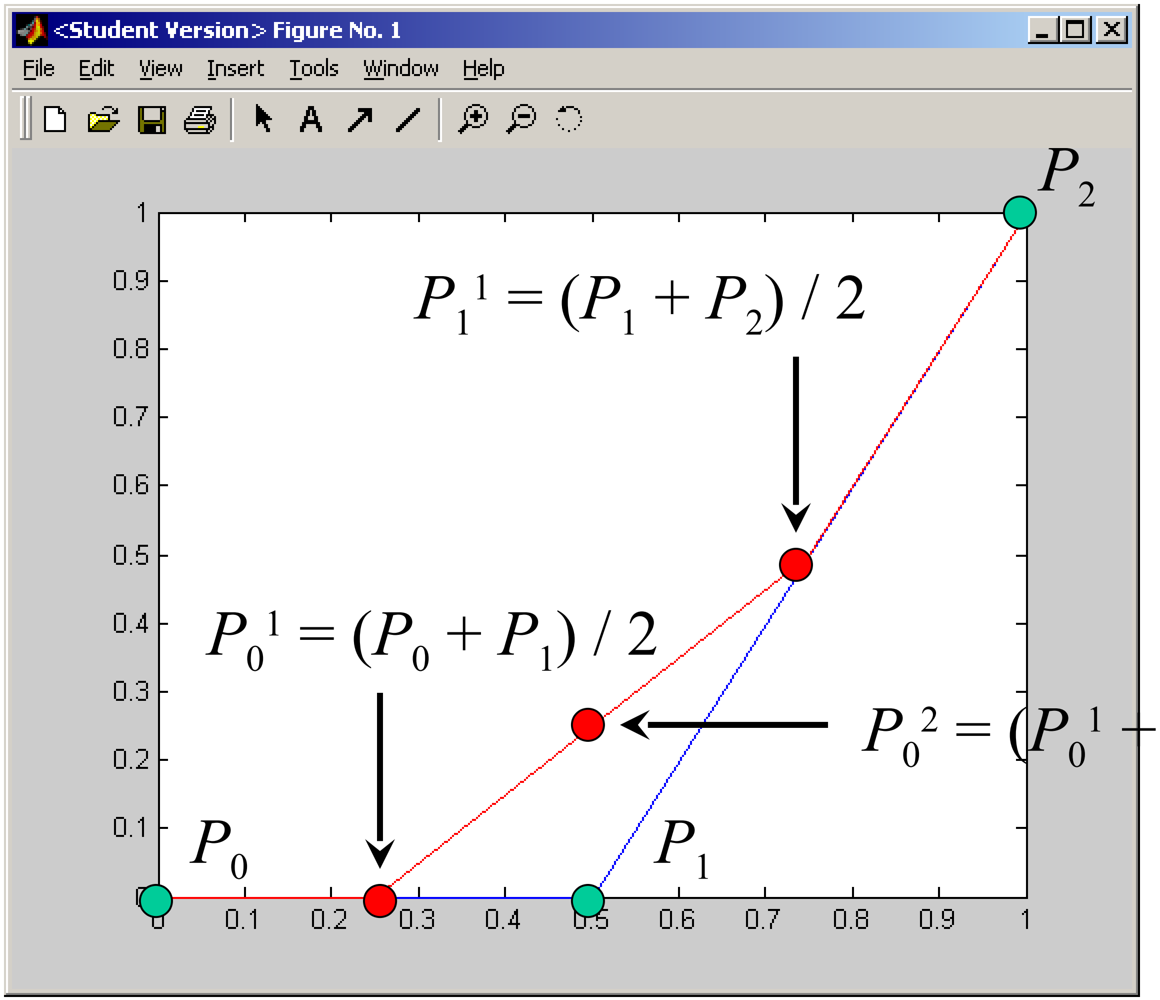
Polynomial Functions

- Want to graph $f(x) = x^2$ over [0, 1]
- We'll use *subdivision*.
- de Casteljau Algorithm, Bézier curve

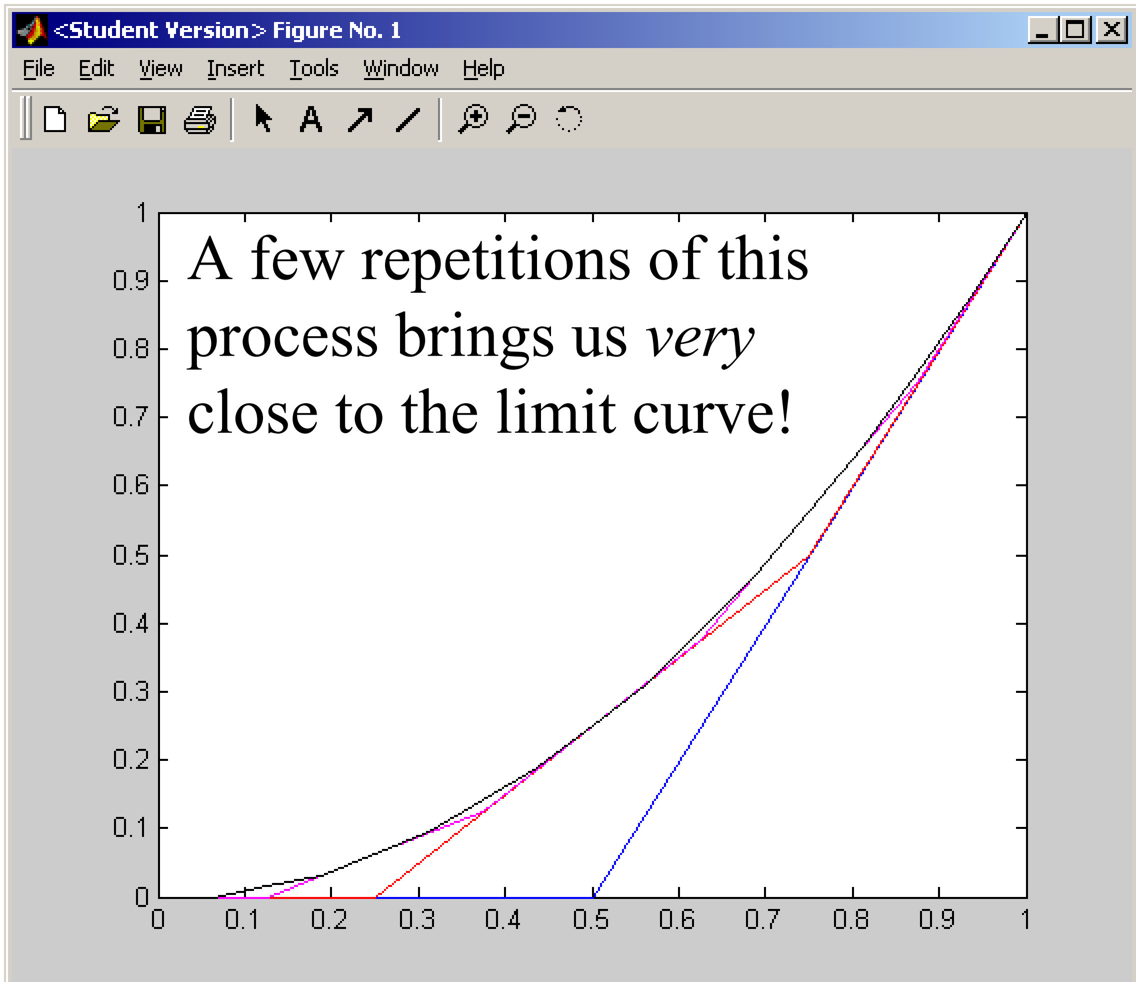
Polynomial Function Subdivision



Polynomial Function Subdivision



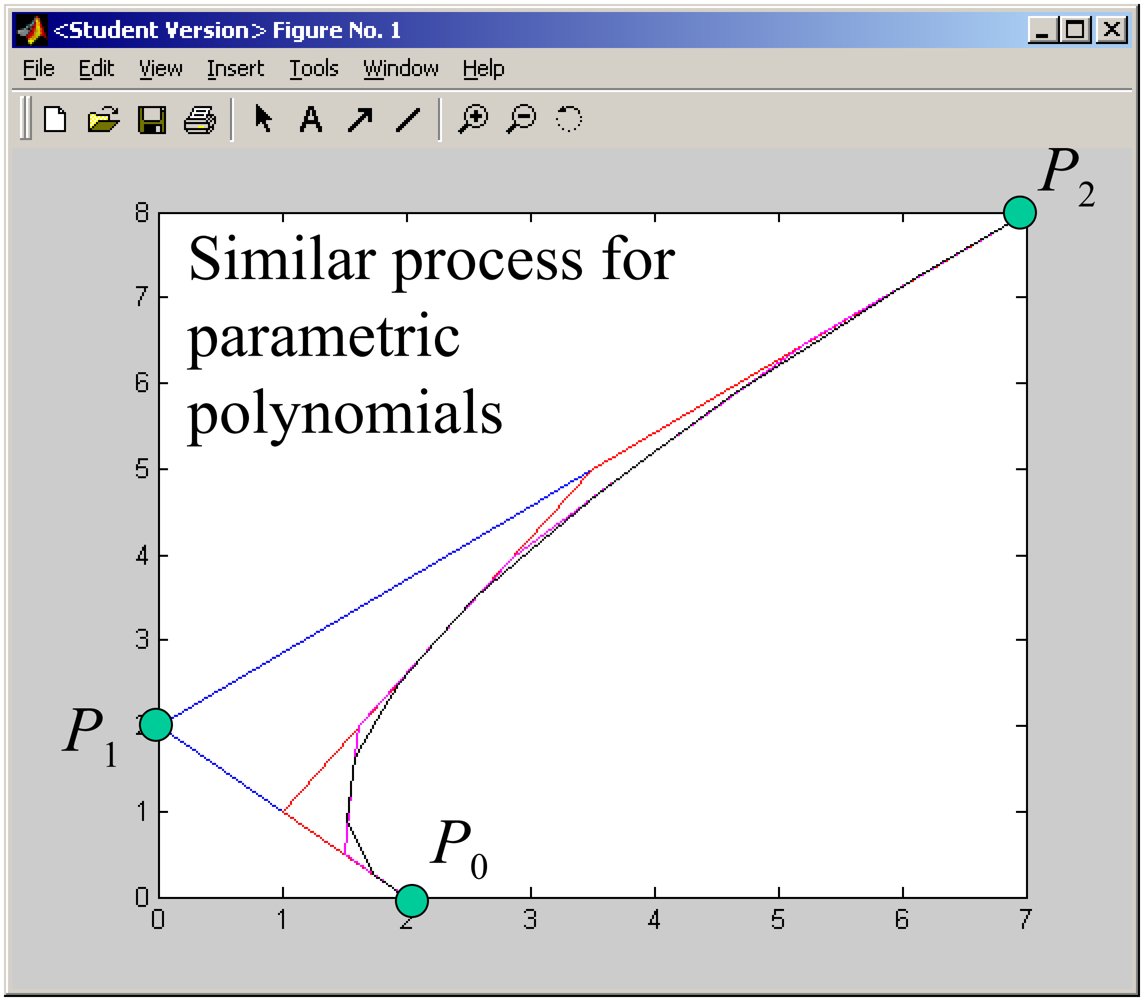
Polynomial Function Subdivision



Complexity of Subdivision

- $O(2^N d^2)$ where:
- Subdivision is iterated N times
- Curve has degree d

Parametric Polynomials



Parametric Rationals

- Projection of a polynomial curve in $\mathbf{R}^3$
- Subdivision algorithm: Subdivide polynomial curve in $\mathbf{R}^3$ and project down to $\mathbf{R}^2$ (Rational de Casteljau Algorithm)

How can we describe rationals?

- Rational parametric form
  - $(u(t)/w(t), v(t)/w(t))$, where u, v, w polynomials
- Rational implicit form
  - $f(x, y) = 0$, where f polynomial

Parametric vs. Implicit

- Implicit form
  - Hard, slow to render
  - Easy to compute with (offset, intersection)
- Parametric form
  - Easy, fast to render
  - Harder to compute with
- Rationals: both forms, both advantages

Algebraic Plane Curves

- Implicit polynomial $f(x, y) = 0$
- Want to subdivide algebraic plane curves.
- How?
- Idea: find rational parameterization!

Rational Parameterization

- Not all algebraic curves are rational.
- How do we tell if a given curve is rational?

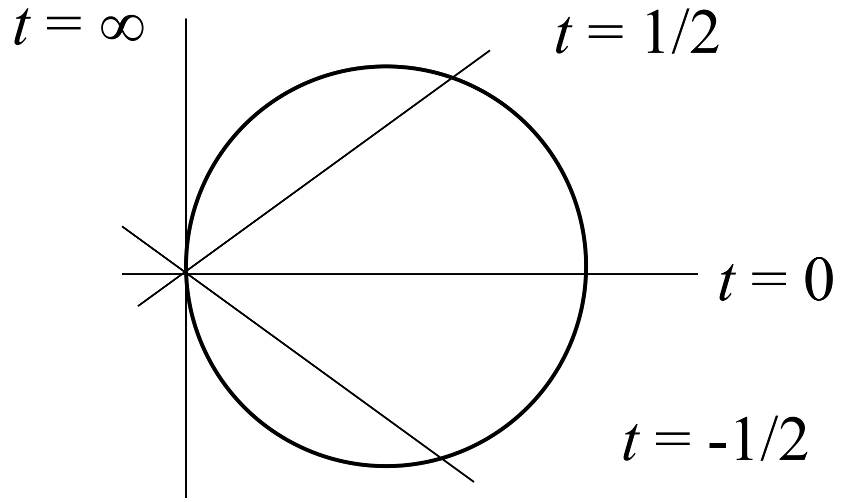
- Cayley-Riemann Theorem:
  - *Genus* $= 0 \Leftrightarrow$ curve is rational

- Compute genus: if genus = 0, it is rational!

Rational Parameterization

- So, given a rational algebraic curve:

- Find rational parameterization with pencil of lines



Complexity of Parameterization

- $O\,(d^{6}\log^{3} d + d^{2}T(d^{2}))$
- $T(d) = O(d^{3}\log^{2} d + d^{2}\log d \log(1/\varepsilon))$
- $T(d)$ is the time to solve one-variable implicit polynomial equation of degree d with precision ε

We ignored the irrational ones!

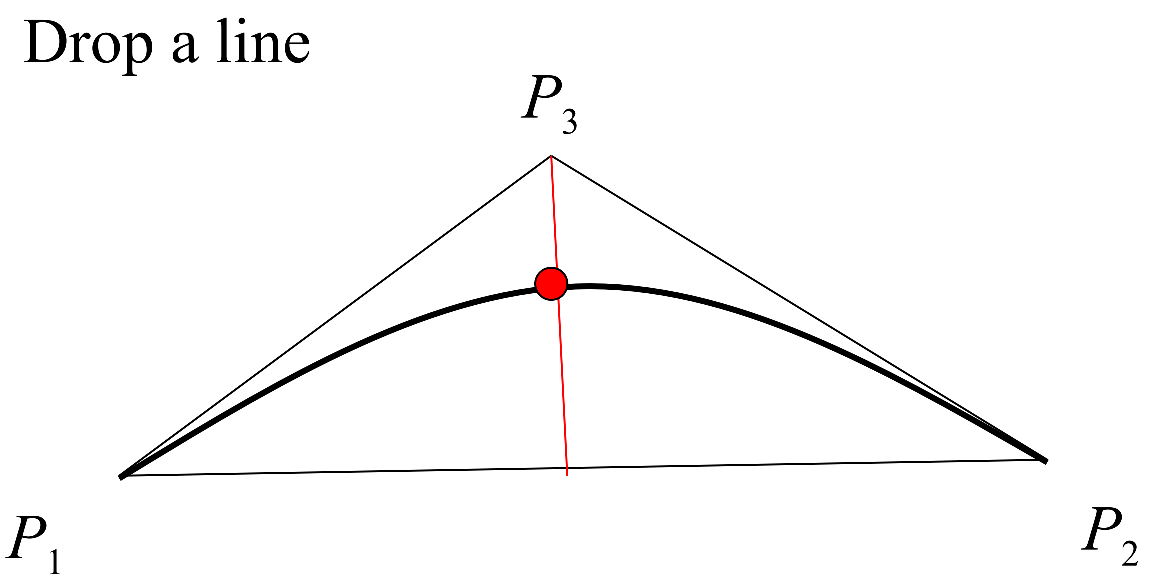
- Let's find a more general technique.

- Planar, triangular *A-splines*:
  - Algebraic plane curves
  - Restricted to a triangle $\Delta P_1 P_2 P_3$
  - Curve passes through P_1 and P_2

Convex A-Splines

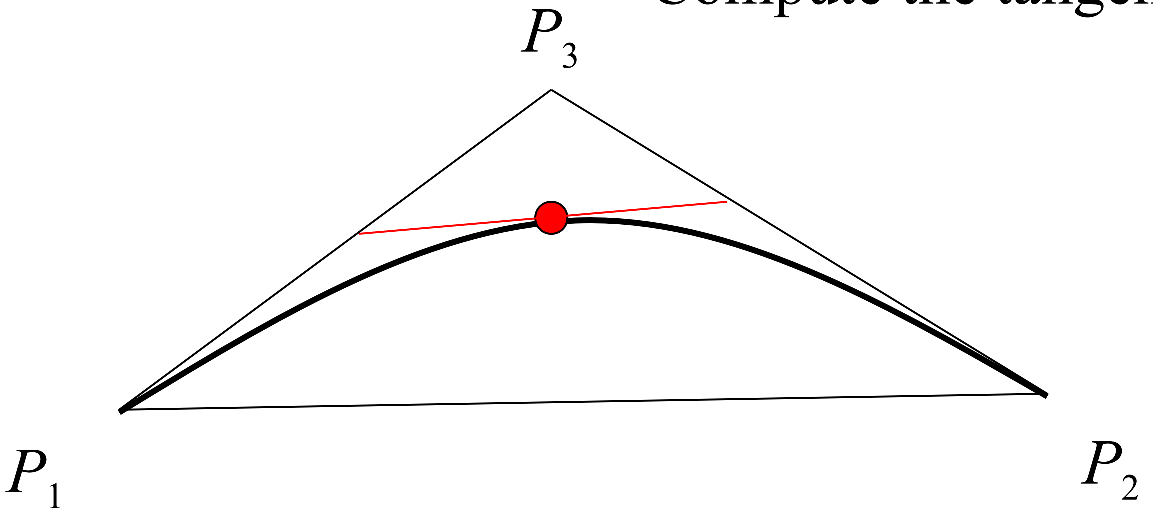
An A-spline is *convex* if it has no inflection points within its triangle.

Subdividing Convex A-Splines



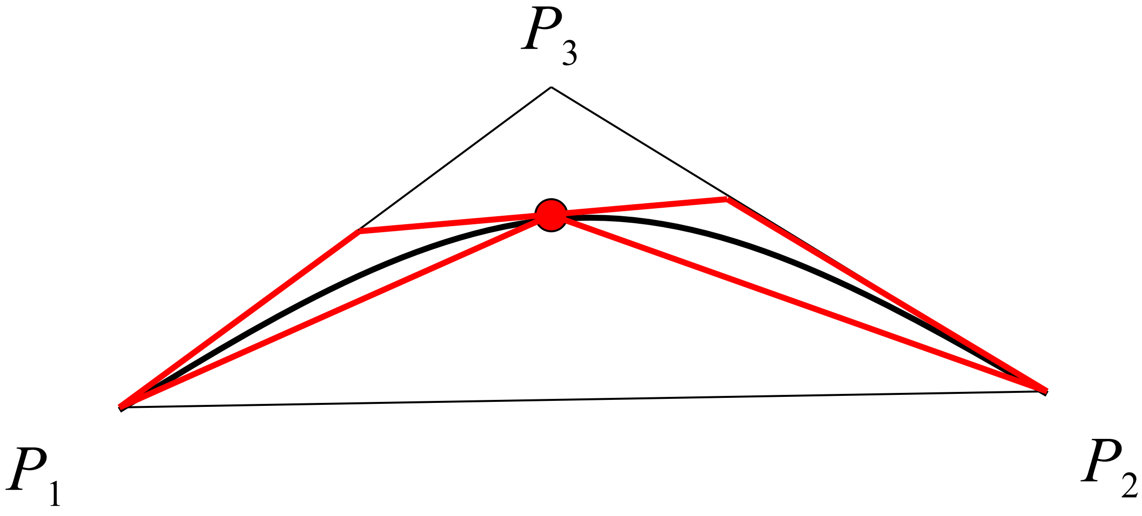
Subdividing Convex A-Splines

Compute the tangent



Subdividing Convex A-Splines

We have two new triangles!
One step of subdivision is complete!



Repeat to subdivide more.

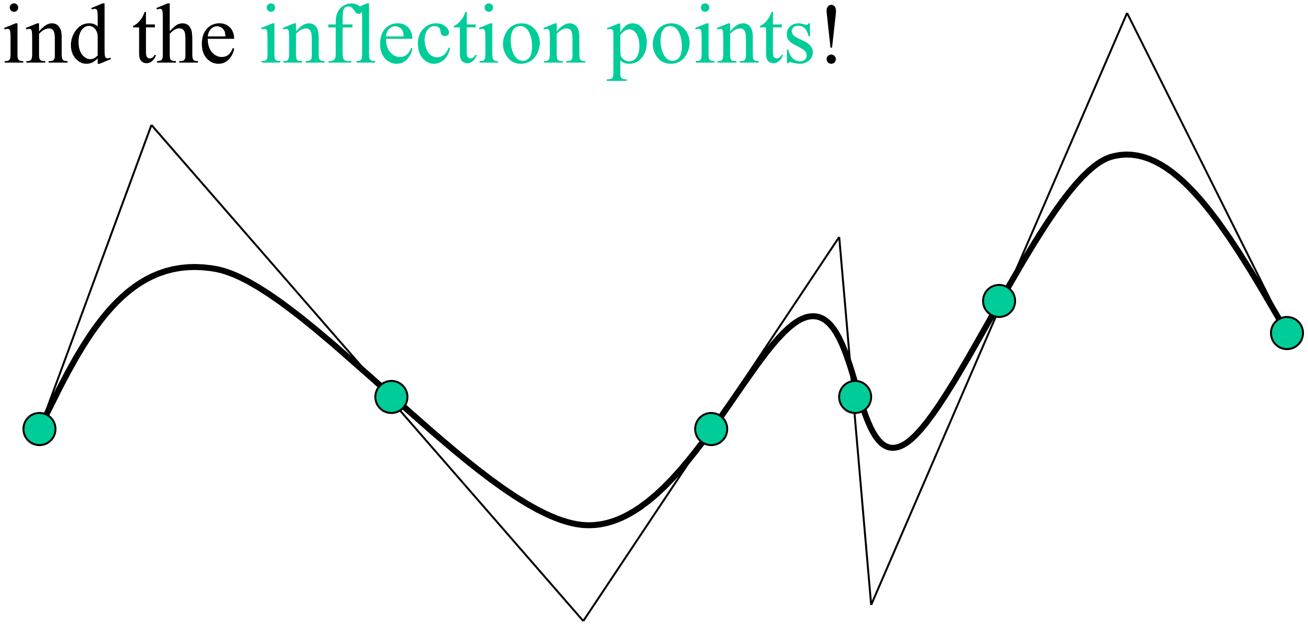
Subdividing A-Splines

- How would we subdivide A-splines which are not convex?

- How about:
  - Divide A-spline into convex pieces
  - Subdivide each piece!

Subdividing A-Splines

- How do we divide an A-spline into convex pieces?
- Find the inflection points!



Finding Inflection Points

- Compute Hessian:

$$H(x, y) = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = 0$$

- Solve $f(x, y) = H(x, y) = 0$
- Theorem:
  - solutions = {inflection points, singularities}

Finding Inflection Points

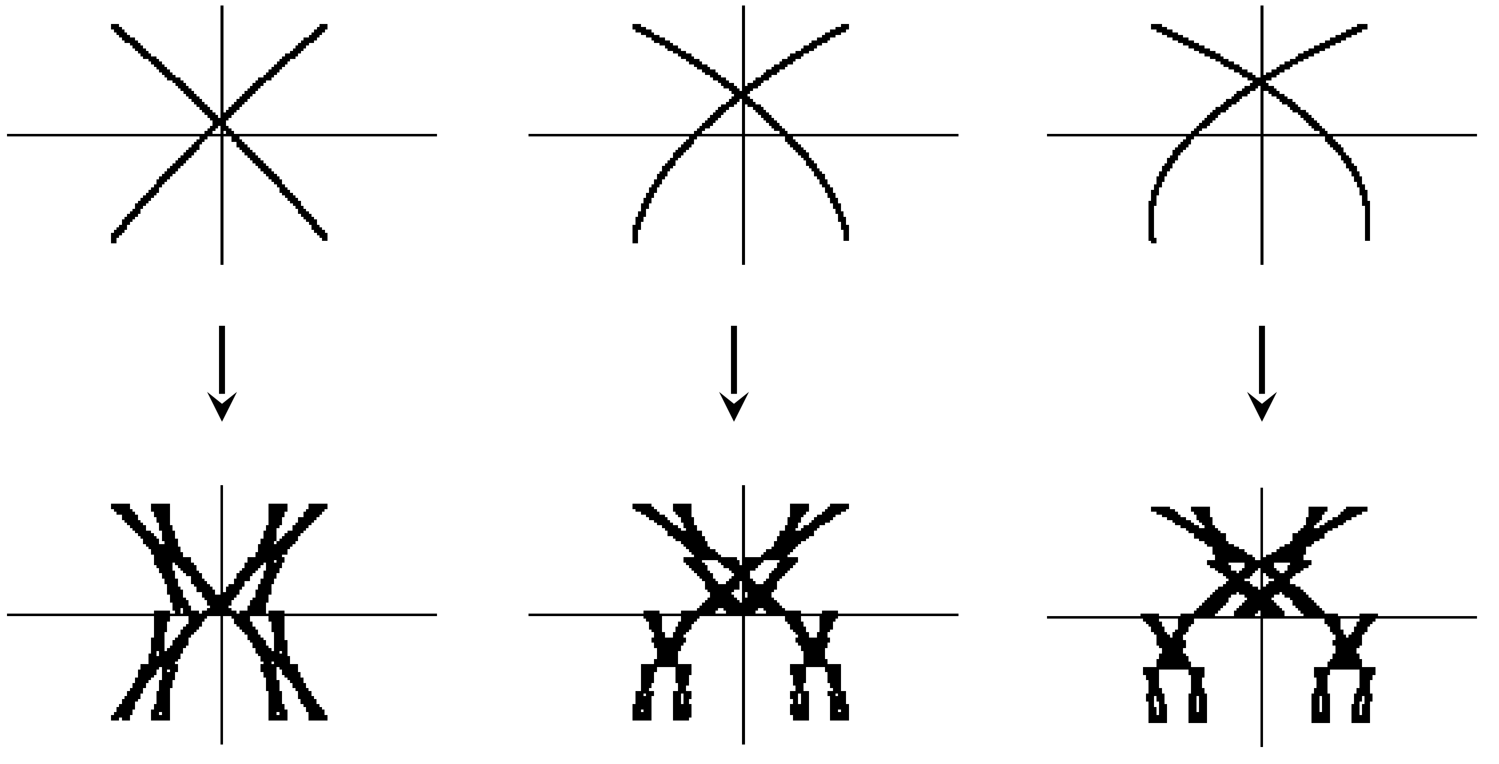
- Compute singularities:
  $f(x, y) = f_x(x, y) = f_y(x, y) = 0$
- Subtract singularities from previous solution:
  {inflections, singularities} – {singularities} = {inflections}!
- Thus we can subdivide *all* A-splines!
- $O\ (2^N d^6 \log^3 d + 2^N d^2 T(d) + T(d^2))$

Further Study: A-Patches

- How about surfaces?

- A-patches (algebraic patches):
  - $f(x, y, z) = 0$ in compact domain
  - Surface of revolution subdivision is already implemented in GANITH
  - Sweeping, lofting A-splines is next

Further Study: Control Fractals



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