Searching for Subdivision Algorithms for Algebraic Curves and Surfaces

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Abstract

In this report we describe known subdivision algorithms for algebraic splines, brainstorm approaches for more finding more efficient subdivision algorithms, and briefly introduce algebraic surface patches, for which efficient subdivision algorithms will likely be more difficult to derive.

1 Introduction

Algebraic curves and surfaces are used extensively in computer graphics visualization. Images are often comprised of these entities because they are easy to describe and manipulate. Low-degree algebraic curves and surfaces are also efficient to render. However, as the degree of an algebraic curve or surface increases, the rendering time increases substantially, using currently known algorithms. In this report we survey some of the currently known techniques of rendering these objects and attempt to describe new approaches which may lead to better rendering algorithms.

All of the rendering algorithms we will look at will be based on the concept of subdivision. Subdivision is based on the idea that for any curve or surface, there is some iterative process which converges quickly to that curve or surface. Usually one step of this iterative process will contain some form of "subdividing" the curve or surface into pieces. So a subdivision algorithm is really just a recursive algorithm which, when applied iteratively to some simple initial set of points, will converge to the desired curve or surface. Ideally a small number of iterations or computations would be needed to render a curve or surface by subdivision.

Efficient subdivision algorithms are known for graphs of rational functions and rationally parameterized curves and surfaces. Also these algorithms can be extended to efficiently subdivide piecewise rational curves and surfaces. However, for implicit curves and surfaces, a direct, efficient subdivision method is not known. We wish to develop such a subdivision algorithm. For simplicity we restrict our attention to *algebraic* splines (curves) and patches (surfaces), where the implicit equations contain only polynomial terms.

2 What is Subdivision?

Consider a two-dimensional surface A sitting in three-dimensional space. Suppose that we wish to render an image of A. We attempt to accomplish this task by finding a small simple set A_0 and a function S which define a sequence

$$A_0, A_1 = S(A_0), A_2 = S(A_1), \cdots$$

such that

$$\lim_{n \to \infty} A_n = A.$$

We wish for the sequence $\{A_i\}$ to converge quickly to A, i.e., we would like A_N to be visually indistinguishable from A where N is small.

3 What Can We Subdivide (Efficiently)?

Efficient subdivision algorithms are known for many special classes of curves and surfaces. Graphs of polynomial functions, parametric polynomial curves, piecewise parametric polynomials, rational parametric polynomials, and piecewise parametric rationals form a hierarchy of classes of curves for which a common subdivision technique exists.

Fast subdivision algorithms also exist for the analogous surfaces: graphs of bivariate rational functions, and parametric and piecewise parametric rational surfaces. We describe subdivision algorithms for rational curves and surfaces below.

Efficient subdivision algorithms are known for quadratic and cubic algebraic curves, as well as tetrahedral and prism patches [10].

But for general algebraic curves and surfaces, efficient subdivision methods are not known.

3.1 Subdivision of Rational Curves

Assume we are given a rational plane curve represented by two rational functions $x(t)$ and $y(t)$, where $t \in [0, 1]$. We can find *control points* $P_0, \ldots, P_n \in \mathbb{R}^2$ and *weights* $w_0, \ldots, w_n \in \mathbb{R}$ such that

$$P(t) = \frac{\sum_{i=0}^{n} w_i P_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}$$

parameterizes the curve, where n is the degree of the largest exponent of t in $x(t)$ or $y(t)$ and

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, i = 0, \ldots, n.$$

Here we have represented the curve as a *rational Bézier curve* [12]. Define

$$P_i^r(t) = (1-t)\frac{w_i^{r-1}(t)}{w_i^r(t)} P_i^{r-1}(t) + t\frac{w_{i+1}^{r-1}(t)}{w_i^r(t)} P_{i+1}^{r-1}(t)$$

and

$$w_i^r(t) = (1-t)w_i^{r-1}(t) + t w_{i+1}^{r-1}(t),$$

where $r = 1, \ldots, n$, $i = 0, \ldots, n-r$, and $P_i^0(t) = P_i$. We may subdivide the curve $P(t)$ into two curves $P([0, u])$ and $P([u, 1])$ where $0 < u < 1$. $P([0, u])$ has control points $P_0^i(u)$ and weights $w_0^i(u)$ for all $i = 0, \ldots, n$. Similarly $P([u, 1])$ has control points $P_{n-i}^i(u)$ and weights $w_{n-i}^i(u)$. This process of subdivision may be iterated to render the target curve efficiently and accurately. For rational space curves given by $x(t)$, $y(t)$, and $z(t)$, the same technique works.

3.2 Subdivision of Rational Surfaces

Suppose we are given a rational surface specified by

$$(x(s, t), y(s, t), z(s, t)),$$

where x, y, and z are rational in $s, t \in [0, 1]$. Then we can compute a *control net* $\{P_{ij}\}_{i,j=0}^{i,j=m,n}$ and weights $\{w_{ij}\}_{i,j=0}^{i,j=m,n}$ so that we parameterize the *rational Bézier* surface by

$$P(s,t) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} w_{ij} P_{ij} B_i^m(s) B_j^n(t)}{\sum_{i=0}^{n}\sum_{j=0}^{n} w_{ij} B_i^m(s) B_j^n(t)},$$

for some m, n. To subdivide this surface patch, simply treat each row of the control net as the control points of a rational Bézier curve. Now use the subdivision algorithm for rational curves on each row of control points to obtain a new set of control points for each row. Combining all of these new control points yields a new subdivided control net for the rational surface. For more on rational Bézier and rational B-spline surfaces, see [12].

4 What is an A-Spline?

An algebraic curve of degree n over a domain D in two-dimensional space is a set of points $(x, y) \in D$ which satisfy a degree-n polynomial equation $G_n(x, y) = 0$. An algebraic spline, or A-spline, of degree n is an algebraic curve whose domain D is a triangle in the plane [6] [10].

Let Q_1, Q_2, and Q_3 be noncollinear points in the two-dimensional space. Let the triangle $\triangle Q_1Q_2Q_3$ be the domain of an A-spline of degree n defined by $G_n(x, y) = 0$. A result from mathematics states that G_n can be replaced by a function F_n over the unit square which is in Bernstein-Bézier form:

$$\begin{aligned} F_n(\alpha_1, \alpha_2) &= \sum_{i+j+k=n} b_{ijk} B_{ijk}^n(\alpha_1, \alpha_2) \\ &= \sum_{i+j+k=n} \frac{n!}{i!j!k!} b_{ijk} \alpha_1^i \alpha_2^j (1 - \alpha_1 - \alpha_2)^k = 0 \end{aligned}$$

We can think of an A-spline as follows. Consider the surface in 3D formed by the points $(x, y, G_n(x, y))$ for all $(x, y) \in \triangle Q_1Q_2Q_3$. Now consider $\triangle Q_1Q_2Q_3$ as a subset of three-dimensional space lying in the xy-plane. The A-spline represented by G_n and $\triangle Q_1Q_2Q_3$ is simply the intersection of the surface and the triangle. So if we apply a polynomial deformation G_n to the domain triangle in three-dimensional space and subsequently intersect it with the domain triangle itself we get an A-spline.

However we would like an A-spline to obey certain rules for our convenience. If we just arbitrarily pick a degree-n polynomial G_n and a domain triangle, we may run into problems. For instance, G_n may have no zeros over the triangle, or worse, $G_n(x, y) = x^2 + y^2 + 1$ has no zeros anywhere. Also we may wish to have a *convex* A-spline, meaning that a single connected piece of the curve lies in the domain triangle, passes through the points Q_1 and Q_2, and has no inflection points within the interior of the domain triangle. An additional common assumption is that the curve is C^1-continuous.

5 How Can We Subdivide A-Splines?

A number of techniques have been developed to subdivide A-splines. Degree 2 and 3 A-splines have fast subdivision algorithms which we describe below [9] [10]. One general technique to subdivide an A-spline is to subdivide the graph of the polynomial function G_n in some neighborhoood of the xy-plane and reduce that to a subdivision of the A-spline. Another technique is to find a rational parameterization for an A-spline [10], if one exists, and then to subdivide the curve using the method for rational curves [12].

5.1 Subdivision by Zero-Contour Representation

Let D be an arbitrary Bernstein-Bézier triangle and let G_n be a polynomial of degree n which defines an A-spline inside D. Consider the graph of G_n over D. This is a parametric polynomial surface given by $\mathbf{\Phi}(s, t) = (s, t, G_n(s, t))$. We can subdivide polynomial surfaces and then we can simply pick out the points with zero as the third coordinate, and we have a rendering of an A-spline. However this method can be slow since it is indirect. We first subdivide a polynomial surface which is efficient. But then picking out each individual point with zero or approximately zero in the third coordinate is inefficient, even if we try to subdivide the polynomial surface in a small neighborhood of the xy-plane.

5.2 Subdivision by Rational Parameterization

Genus is a property of curves and surfaces. The genus of an algebraic curve is a number defined in terms of its degree and its number of singularities. Algebraic curves of genus zero are rationally parameterizable. Thus one technique used to subdivide genus-zero algebraic curves is to simply compute a rational parameterization of the curve and subdivide it as one would a rational curve. For degree 2 and 3 curves there are efficient methods for finding a rational parameterization of an algebraic curve of genus zero and also for testing whether a curve does in fact have genus zero. Here we look at techniques for rationally parameterizing genus zero algebraic curves.

5.2.1 Rational Quadratic Algebraic Curves

Quadratic algebraic curves are also known as conic sections. All conics have genus zero and are thus rationally parameterizable. We show how to find a rational parameterization of an arbitrary conic below. The information below is based on [10].

Given

$$f(x, y) = ax^2 + by^2 + cxy + dx + ey + f = 0, \tag{1}$$

we derive a rational parameterization of the curve defined by this implicit equation. If $b = 0$, then we have

$$\begin{aligned} ax^2 + cxy + dx + ey + f &= 0 \\ ax^2 + dx + f + (cx + e)y &= 0 \\ y &= -\frac{ax^2 + dx + f}{cx + e} \end{aligned}$$

So the curve is parameterized as

$$P(t) = \left(t, -\frac{at^2 + dt + f}{ct + e}\right). \tag{2}$$

What about when $cx + e = 0$? Then any point (x, y) where $ax^2 + dx + f = 0$ is on the curve.

If $a = 0$, a similar parameterization follows. If $a, b \neq 0$, then select an arbitrary point (x_0, y_0) on the curve. Let $L(t)$ denote the line with slope t that passes through (x_0, y_0).

$$L(t) = \{(x, y) : y - y_0 = t(x - x_0)\} = \{(x, y) : y = t(x - x_0) + y_0\}.$$

Now solve the equation

$$f(x, t(x - x_0) + y_0) = 0$$

for x. This yields two solutions. One solution is x_0 and the other solution is the desired parameterization $x(t)$. Plugging into the equation for $L(t)$ yields $y(t) = t(x(t) - x_0)) + y_0$.

5.2.2 Rational Cubic Algebraic Curves

Cubic algebraic curves are not all rationally parameterizable. However we present a way to efficiently determine whether a given cubic algebraic curve has genus zero and if so derive its rational parameterization [9] [10].

Given $f(x, y) = ax^3 + by^3 + cx^2y + dxy^2 + ex^2 + fy^2 + gxy + hx + iy + j = 0$, we want to find rational parameterization *if one exists.* Assume one exists. Substitute:

$$x = x_1 + qy_1, y = y_1$$

so that

$$f(x, y) = g(x_1, y_1) = L(q)y_1^3 + h(x_1, y_1)$$

where

$$L(q) = aq^3 + cq^2 + dq + b$$

and q satisfies $L(q) = 0$.

Now we just need to parameterize $h(x_1, y_1)$. First write

$$h(x_1, y_1) = h_1(x_1)y_1^2 + h_2(x_1)y_1 + h_3(x_1) = 0, \qquad (3)$$

where h_1, h_2, h_3 are degrees 1, 2, 3, respectively. The discriminant of h with respect to y_1 is

$$h_4(x_1) = h_2(x_1)^2 - 4h_1(x_1)h_3(x_1).$$

The curve is rationalizable iff x_1 is a multiple root of h_4. x_1 can be real or complex, we consider only the real case.

Do transformation $y_2 \equiv 2h_1y_1 + h_2$ so that

$$\begin{aligned} 4h_1h &= 4h_1^2y_1^2 + 4h_1h_2y_1 + 4h_1h_3 \\ &= (2h_1y_1 + h_2)^2 - (h_2^2 - 4h_1h_3) \\ &= y_2^2 - h_4. \end{aligned}$$

For any real number r, the Taylor series of h_4 about r is

$$\begin{aligned} h_4(x_1) &= h_4(r) + h_4'(r)(x_1 - r) + h_4''(r)(x_1 - r)^2 + \\ & \quad h_4'''(r)(x_1 - r)^3/6 + h_4''''(r)(x_1 - r)^4/24 \end{aligned}$$

$$= \quad h_4(r) + h_4'(r)(x_1 - r) + q_2(x_1)(x_1 - r)^2, \tag{4}$$

where q_2 is a polynomial of degree 2:

$$q_2(x_1) = h_4''(r)/2 + h_4'''(r)(x_1 - r)/6 + h_4''''(r)(x_1 - r)^2/24.$$

Now define $y_3 \equiv y_2/(x_1 - r)$ into equation 4 together with equation 3:

$$\begin{aligned} 4h_1h &= y_1^2 - h_4(x_1) \\ &= (y_3^2 - q_2(x_1))(x_1 - r)^2 + h_4'(r)(x_1 - r) + h_4(r) \\ &\equiv k(x_1, y_3). \end{aligned}$$

Choose r to be a multiple root of h_4 so that $h_4(r) = h_4'(r) = 0$. Then we have

$$k(x_1, y_3) = (y_3^2 - q_2(x_1))(x_1 - r)^2.$$

Let

$$C(x_1, y_3) \equiv y_3^2 - q_2(x_1).$$

C is an implicit quadratic, which can be parameterized by the methods of the previous section so that $x_1(t)$ and $y_3(t)$ are rational.

Now apply all transformations in reverse. The parameterization for $C(x_1, y_3) = 0$ yields one for $k(x_1, y_3) = 0$. Since $y_2 = y_3(x_1 - r)$ and $y_1 = (y_2 - h_2)/2h_1$, we can parameterize $h(x_1, y_1) = 0$ and thus $g(x_1, y_1) = 0$. Now $x(t) = x_1(t) + qy_1(t)$ and $y(t) = y_1(t)$, so we have the desired parameterization for $g(x_1(t), y_1(t)) = f(x(t), y(t)) = 0$.

5.3 Subdividing Convex A-Splines

In general an A-spline is given by its implicit equation $G_n(x, y) = 0$ over a domain D. We showed earlier that this is equivalent to an equation $F_n(\alpha_1, \alpha_2) = 0$ where the domain of F_n is the unit square. If the A-spline has genus zero, we wish to find a rational parameterization

$$P(t) = \frac{\sum_{i=0}^{n} w_i P_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \tag{5}$$

where $P_i \in \mathbb{R}^2$, $w_i \in \mathbb{R}$, and $B_i^n(t)$ are the used earlier to define rational Bézier curves. We can modify the weights so that, without of loss of generality, $w_0 = w_n = 1$ [6], [12]. We now explore rational parameterization of convex C^1-continuous quadratic and cubic A-splines. The Bernstein-Bézier coefficients of a C^1 A-spline must satisfy $b_{n00} = b_{0n0} = b_{(n-1)01} = b_{0(n-1)1} = 0$ [10].

5.3.1 Rational Parameterization of Quadratic A-Splines

A rational parameterization of a convex C^1 quadratic A-spline would have the form

$$P(t) = \frac{Q_0 B_0^2(t) + w_1 Q_2 B_1^2(t) + Q_1 B_2^2(t)}{B_0^2(t) + w_1 B_1^2(t) + B_2^2(t)}, \tag{6}$$

if the A-spline interpolates Q_0 and Q_1. As mentioned in [10], if

$$w_1 = \sqrt{-\frac{b_{110}}{2b_{002}}} \geq 0,$$

then a quadratic A-spline defined as aboveis rationally parameterizable. Equation 6 is said to represent a 2/2 rational parameterization of a quadratic A-spline.

5.3.2 Rational Parameterization of Cubic A-Splines

Suppose $F_3(\alpha_1, \alpha_2) = 0$ represents a C^1 convex cubic A-spline in the domain triangle (Q_0, Q_1, Q_2). There is no 2/2 rational parameterization for a nondegenerate cubic A-spline, but there is a method to derive a 3/3 rational parameterization which has the form

$$P(t) = \frac{P_0 B_0^3(t) + w_1 P_1 B_1^3(t) + w_2 P_2 B_2^3(t) + P_3 B_3^3(t)}{B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + B_3^3(t)}, \tag{7}$$

where

$$\begin{aligned} P_0 &\equiv Q_0 \\ P_1 &\equiv (1-\alpha)Q_0 + \alpha Q_2 \\ P_2 &\equiv (1-\beta)Q_1 + \beta Q_2 \\ P_3 &\equiv Q_1, \end{aligned}$$

and α, β, w_1, and w_2 are parameters to be determined [10].

5.4 Subdividing Nonconvex A-Splines

A topic which deserves further study is the subdivision of nonconvex A-splines. These are A-splines which have a sequence of inflection points. To subdivide such an A-spline, we simply break it up into a sequence of A-splines where each new domain triangle contains the part of the A-spline between an inflection point and an endpoint or another inflection point. At each inflection point, a side of the domain triangle of one new A-spline and a side of the next A-spline's domain triangle form a tangent line segment to the A-spline. There is a known efficient method for doing this.

The next step is to find a clever fast way to subdivide each of the individual A-splines that were created in the process above. All of these A-splines are now convex. We desire to find a fast algorithm for computing a tangent to a convex A-spline such that the domain triangle of that A-spline can be subdivided into two new domain triangles. Then this process can be repeated, individually for each A-spline, to yield a fast subdivision algorithm for the entire nonconvex A-spline. Ideas for performing this are presented in [14].

5.5 Searching for New Subdivision Algorithms

Let us look back at Section 3. What is subdivision? Subdivision in a general sense is the concept of starting with an initial set A_0 and a function S such that when S is repeatedly applied to A_0, a sequence forms which converges to a desired target set A. Ideally we would like this process to converge quickly to A, i.e., for small N, we would like A_N as defined in Section 3 to be close enough to A so that further iteration is impractical or unnecessary due to the limit of the pixel width.

The paper [13] introduces a slightly unconventional way of looking at the subdivision of polynomial curves. Instead of describing the subdivision as a sequence of simple linear combinations, the paper describes the process as the successive combined iteration of two affine maps (which comprise the subdivision function S) applied to any initial compact set A_0. Looking into different kinds of maps other than affine ones, we may be able to describe subdivision processes for more complex curves, including algebraic curves.

6 What is an A-Patch and How Can We Hope to Subdivide A-Patches?

An *algebraic surface* of degree n is the set of points in three-dimensional space satisfying $G_n(x, y, z) = 0$ where G_n is a polynomial function of degree n. An algebraic surface defined over a compact domain is an *algebraic surface patch*, or A-patch. Let Q_1, Q_2, Q_3, Q_4 be noncoplanar points in three-dimensional space. These points are vertices of a tetrahedron V. Now by the Bernstein-Bézier representation [5], since the vertices of V are affinely independent, for each $p \in V$, there is a unique point $(\alpha_1, \alpha_2, \alpha_3)$ in the unit cube such that

$$p = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3 + (1 - \alpha_1 - \alpha_2 - \alpha_3) Q_4.$$

Furthermore, there exist real numbers, called *control points*, b_{ijkl} such that

$$\begin{aligned} 0 = G_n(x, y, z) &= F_n(\alpha_1, \alpha_2, \alpha_3) \\ &= \sum_{i+j+k+l=n} b_{ijkl} B_{ijkl}(\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

where

$$B_{ijkl}(\alpha_1, \alpha_2, \alpha_3) = \frac{n!}{i!j!k!l!} b_{ijkl} \alpha_1^i \alpha_2^j \alpha_3^k (1 - \alpha_1 - \alpha_2 - \alpha_3)^l.$$

An A-patch does not have to be defined inside a tetrahedron. For each of the xy-, yz-, and xz-planes, we can have either a tensor (square) or barycentric (triangular) domain, conventionally. This yields possibilities for domains to be shapes such as cubes and pyramids, as well as tetrahedra. The above information and more details are explained in [7].

Subdivision techniques exist for A-patches via rational parameterization in two variables. Methods for parameterizing parameterizable prism and tetrahedral patches, as well as ways of approximating A-patches with triangular rational surfaces, are explained in [10].

7 Conclusion

Efficient subdivision algorithms for curves and surfaces describable by rational functions and parameterizations are generally known. But once one crosses into the world of implicit algebraic curves and surfaces, less is known about fast subdivision for these objects. However, with what is known about rational parameterization of certain implicit curves and surfaces, plus other innovative ideas on converging sequences of geometric objects, there is hope for finding more efficient and perhaps more direct methods for subdividing implicit curves and surfaces.

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